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On the Henstock-Stieltjes Integral for ℓ_p-Valued Functions, $0 < p < 1$

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ABSTRACT

In this paper, the Henstock-Stieltjes integral (HS integral) of a function with values ranging in an ℓ_p−space, with $0 < p < 1$, is defined. Results such as the uniqueness of the HS integral, the Cauchy-Criterion for HS-integrability, HS-integrability of continuous functions, linearity of the HS-integral, HS-integrability of a function over a subinterval, and the additive property of HS integral on subintervals are obtained.

Keywords: δ-fine tagged partition, Henstock-Stieltjes integral, quasi-norm, quasi-Banach spaces, ℓ_p-space

1 Introduction

It is well known that the Riemann integral is not adequate for advanced mathematics, since there are many functions that are not Riemann integrable. To overcome these deficiencies, Henri Lebesgue developed his integral around 1902, and his integral has become the suitable integral for almost mathematical researches. However, there are also difficulties with the Lebesgue integral. In 1950's, J. Kurzweil introduced a generalized version of the Riemann integral. Its definition is Riemann-like, but was recognized even much powerful than the Lebesgue integral.

Further studies have been carried out to make some generalizations of Henstock integral. Cao (1992) generalized the definition of the Henstock integral for real-valued functions to functions taking values in Banach spaces and investigate some of its properties. Several authors have studied Henstock integral (Lim et al., 1998; Solodov, 1999; Boccuto and Sambucini, 2004).

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2 Preliminaries

In this paper, we consider the integrals of functions defined on a closed interval $[a, b]$ and ranging in a quasi-Banach ℓ_p-space $(0 < p < 1)$ which carries a quasi-norm denoted by $|| \ ||_p$.

Definition 2.1. A **tagged partition** of $[a, b] \subset \mathbb{R}$ is a finite collection

$$P = \{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\},$$

where the $[x_{i-1}, x_i]$ are pairwise non-overlapping subintervals of $[a, b]$, with $\bigcup_{i=1}^{n}[x_{i-1}, x_i] = [a, b]$ and $t_i \in [x_{i-1}, x_i]$ are the **tags.**

Definition 2.2. Let δ be a positive function defined on the interval $[a, b]$. A tagged partition $P = \big\{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\big\}$ is δ-**fine** if

$$[x_{i-1}, x_i] \subset \big(t_i - \delta(t_i), t_i + \delta(t_i)\big).$$

for each $i = \{1, 2, \ldots, n\}$.

Now, we give the definition of Henstock-Stieltjes integral for ℓ_p-valued functions.

Definition 2.3. Let α be an increasing real-valued function on $[a, b]$. A function $f : [a, b] \to \ell_p$ is **Henstock-Stieltjes integrable (HS-integrable)** with respect to α on $[a, b]$ if there exists $A \in \ell_p$ so that for every $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that for every $\delta-$fine tagged partition $P = \{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\}$ of $[a, b]$, we have $\big|\big|S(f, \alpha; P) - A\big|\big|_p < \varepsilon$ where

$$S(f, \alpha; P) = \sum_{i=1}^{n} f(t_i)\Big[\alpha(x_i) - \alpha(x_{i-1})\Big].$$

We call A the **Henstock-Stieltjes integral (HS integral)** of f with respect to α on $[a, b]$ and write $\int_a^b f d\alpha = A$.

3 Basic Properties

Throughout this section, we let $f : [a, b] \to \ell_p$ and α be an increasing real-valued function on $[a, b]$. The first theorem guarantees the uniqueness of Henstock-Stieltjes integral of an ℓ_p-valued function, if it exists.

Theorem 3.1. A function f has at most one HS integral with respect to α on $[a, b]$.

Proof. Suppose that f is HS-integrable with respect to α on $[a,b]$ and A_1 and A_2 are HS integrals of f with respect to α on $[a,b]$ with $A_1 \neq A_2$. Let $\varepsilon > 0$. Then there exist positive functions δ_1 and δ_2 such that for each δ_1-fine tagged partition P_1 of $[a,b]$ and for each δ_2-fine tagged partition P_2 of $[a,b]$, we have

$$||S(f,\alpha;P_1) - A_1||_p < \frac{\varepsilon}{2} \ \ and \ \ ||S(f,\alpha;P_2) - A_2||_p < \frac{\varepsilon}{2},$$

respectively. Define a positive function δ on $[a,b]$ by $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$. Let P be any $\delta-$fine tagged partition of $[a,b]$ and let $\varepsilon = \frac{||A_1-A_2||_p}{2^{\frac{1}{p}}}$. Then,

$$\begin{aligned} \big|\big|A_1 - A_2\big|\big|_p &\leq 2^{\frac{1}{p}}\Big(\big|\big|A_1 - S(f,\alpha;P)\big|\big|_p + \big|\big|A_2 - S(f,\alpha;P)\big|\big|_p\Big) \\ &= 2^{\frac{1}{p}}\varepsilon = ||A_1 - A_2||_p, \end{aligned}$$

which is a contradiction. Thus, $A_1 = A_2$. □

In the next theorem, we give the Cauchy criterion for Henstock-Stieltjes integrability of ℓ_p-valued functions.

Theorem 3.2. ***(Cauchy Criterion)*** A function f is HS-integrable with respect to α on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a function $\delta : [a,b] \to \mathbb{R}^+$ such that for all $\delta-$fine partitions $P_1 = \{(t_i, [x_{i-1}, x_i])\}$ and $P_2 = \{(t_i', [x_{i-1}', x_i'])\}$ of $[a,b]$,

$$\Big|\Big|S(f,\alpha;P_1) - S(f,\alpha;P_2)\Big|\Big|_p < \varepsilon.$$

Proof. Suppose that $\int_a^b f d\alpha = A$. Given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that

$$\big|\big|S(f,\alpha;P_1) - A\big|\big|_p < \frac{\varepsilon}{2(2^{\frac{1}{p}})} \ \ and \ \ \big|\big|S(f,\alpha;P_2) - A\big|\big|_p < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_1 and P_2 are $\delta-$fine tagged partitions of $[a,b]$. Now,

$$\begin{aligned} \big|\big|S(f,\alpha;P_1) - S(f,\alpha;P_2)\big|\big|_p &= \big|\big|S(f,\alpha;P_1) - A + A - S(f,\alpha;P_2)\big|\big|_p \\ &\leq 2^{\frac{1}{p}}\left(\big|\big|S(f,\alpha;P_1) - A\big|\big|_p + \big|\big|S(f,\alpha;P_2) - A\big|\big|_p\right) \\ &< \varepsilon. \end{aligned}$$

Thus, the Cauchy Criterion is satisfied.

Suppose the Cauchy criterion holds. Then for each $n \in \mathbb{N}$, there exists a positive function δ_n on $[a,b]$ such that

$$||S(f,\alpha;Q_1) - S(f,\alpha;Q_2)||_p < \frac{1}{n}$$

whenever Q_1 and Q_2 are δ_n-fine tagged partitions of $[a, b]$. Assume without loss of generality that $\delta_n \geq \delta_{n+1}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let P_n be a δ_n-fine tagged partition of $[a, b]$.

Let $\varepsilon > 0$. By Archimedian property, there exists a natural number N such that $\frac{1}{\varepsilon} < N$. Let $N \leq n \leq m$. Then P_n and P_m are δ_n-fine tagged partitions of $[a, b]$. It follows that

$$||(S(f, \alpha; P_n) - S(f, \alpha; P_m)||_p < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\big(S(f, \alpha; P_n)\big)_{n=1}^{\infty}$ is Cauchy.

Hence, the sequence $\big(S(f, \alpha; P_n)\big)_{n=1}^{\infty}$ converges, say to A. Let $\varepsilon > 0$. Then there exists N_1 such that for all $n \geq N_1$,

$$||S(f, \alpha; P_n) - A||_p < \frac{\varepsilon}{2^{\frac{p+1}{p}}}.$$

By Archimedian property, there exists N_2 such that $\frac{2^{\frac{p+1}{p}}}{\varepsilon} < N_2$. Now, let $N_0 = max\{N_1, N_2\}$ and $n \geq N_0$. Let $\delta = \delta_{N_1}$ and P be a δ-fine tagged partition of $[a, b]$. Hence,

$$\big|\big|S(f, \alpha; P) - S(f, \alpha; P_{N_1})\big|\big|_p < \frac{1}{n} < \frac{1}{N_0} \leq \frac{1}{N_2} < \frac{\varepsilon}{2^{\frac{p+1}{p}}}.$$

Thus,

$$\begin{aligned}
\big|\big|S(f, \alpha; P) - A\big|\big|_p &= \big|\big|S(f, \alpha; P) - S(f, \alpha; P_{N_1}) + S(f, \alpha; P_{N_1}) - A\big|\big|_p \\
&\leq 2^{\frac{1}{p}}\Big(\big|\big|S(f, \alpha; P) - S(f, \alpha; P_{N_1})\big|\big|_p + \big|\big|S(f, \alpha; P_{N_1}) - A\big|\big|_p\Big) \\
&< 2^{\frac{1}{p}}\left(\frac{\varepsilon}{(2^{\frac{p+1}{p}})} + \frac{\varepsilon}{(2^{\frac{p+1}{p}})}\right) = \varepsilon.
\end{aligned}$$

□

Theorem 3.3. If a function f is continuous and α is of bounded variation, then f is HS-integrable with respect to α on $[a, b]$.

Proof. Let $\varepsilon > 0$. Since α is of bounded variation, there exists $M > 0$ such that

$$\sum_{i=1}^{n} |\alpha(x_{i-1}) - \alpha(x_i)| \leq M$$

for any partition P of $[a, b]$. Since f is continuous on $[a, b]$, f is also uniformly continuous on $[a, b]$; that is, there exists $\Delta > 0$ such that if $|x - y| < \Delta$ for $x, y \in [a, b]$, then

$$||f(x) - f(y)||_p < \frac{\varepsilon}{M2^{\frac{n-1}{p}}}.$$

Now, define a positive function δ on $[a, b]$ by $\delta(x) = \frac{\Delta}{2}$ for all $x \in [a, b]$ and let P_1 and P_2 be $\delta-$fine partitions of $[a, b]$. We need to show that

$$||S(f, \alpha; P_1) - S(f, \alpha; P_2)||_p < \varepsilon.$$

Without loss of generality, suppose further that P_1 is a refinement of P_2. Thus, it sufficient to show the following two cases:
Case 1. We first consider the case where the tagged partitions P_1 and P_2 differs only in their tags, that is, we have $P_1 = \{(t_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\}$ and $P_2 = \{(t'_i, [x_{i-1}, x_i]) : i = 1, 2, \ldots, n\}$ where $t_i \neq t'_i$ for each i. Let $d_i = \alpha(x_{i-1}) - \alpha(x_i)$. Hence,

$$\begin{aligned}
||S(f,\alpha; P_1) - S(f, \alpha; P_2)||_p &= \Big|\Big| \sum_{i=1}^{n} f(t_i)d_i - \sum_{i=1}^{n} f(t'_i)d_i \Big|\Big|_p \\
&\leq 2^{\frac{n-1}{p}} \sum_{i=1}^{n} ||(f(t_i)d_i - f(t'_i)d_i)||_p \ < \ 2^{\frac{n-1}{p}} \sum_{i=1}^{n} \Big[|d_i| \Big(\frac{\varepsilon}{M2^{\frac{n-1}{p}}} \Big) \Big] \\
&\leq \Big(\frac{\varepsilon}{M} \Big) (M) = \varepsilon.
\end{aligned}$$

Case 2. Suppose that P_1 is obtained from P_2 by inserting one point, say $x_j^{'}$ with $x_j < x_j^{'} < x_{j+1}$ where $x_j^{'}$ is the intermediate point for the two intervals $[x_j, x_j^{'}]$ and $[x_j^{'}, x_{j+1}]$ of P_1. Hence,

$$\begin{aligned}
||S(f,\alpha; P_1) - S(f, \alpha; P_2)||_p &= \Big|\Big| f(t'_j)[\alpha(x'_j) - \alpha(x_j)] \\
&\quad + f(t_{j+1})[\alpha(x_{j+1}) - \alpha(x'_j)] - f(t_{j+1})[\alpha(x_{j+1}) - \alpha(x_j)] \Big|\Big|_p \\
&= \big| [\alpha(x'_j) - \alpha(x_j)] \big| \Big|\Big| (f(t'_j) - f(t_{j+1})) \Big|\Big|_p \\
&< M \left(\frac{\varepsilon}{M2^{\frac{n-1}{p}}} \right) \leq \varepsilon.
\end{aligned}$$

□

We now apply the Cauchy criterion in showing the integrability on a subinterval.

Theorem 3.4. If a function f is HS-integrable with respect to α on $[a, b]$, then f is HS-integrable with respect to α on $[c, d]$ for every subinterval $[c, d]$ of $[a, b]$.

Proof. Let $[c, d] \subseteq [a, b]$ and let $\varepsilon > 0$. By the Cauchy Criterion, there exists a positive function δ on $[a, b]$ such that

$$||S(f, \alpha; P_1) - S(f, \alpha; P_2)||_p < \varepsilon$$

whenever P_1 and P_2 are $\delta-$fine tagged partition of $[a, b]$. Consider the following cases.
Case 1. Suppose $a = c$. Let P_b be a δ-fine tagged partition of $[d, b]$. Also, let P'_1 and P'_2

be δ-fine tagged partitions of $[c,d]$. Define $P = P_1^{'} \bigcup P_b$ and $Q = P_2^{'} \bigcup P_b$. Then P and Q are $\delta-$fine tagged partitions of $[a,b]$. Hence,

$$\begin{aligned} ||S(f,\alpha;P_1^{'}) - S(f,\alpha;P_2^{'})||_p &= ||S(f,\alpha;P_1^{'}) + S(f,\alpha;P_b) \\ &\quad - (S(f,\alpha;P_2^{'}) + S(f,\alpha;P_b))||_p \\ &\leq ||S(f,\alpha;P) - S(f,\alpha;Q)||_p < \varepsilon. \end{aligned}$$

Case 2. Suppose $b = d$. The proof is similar to Case 1.
Case 3. Suppose $[c,d] \subset [a,b]$. The proof follows from Case 1 and Case 2.
Thus, for any cases, f is integrable with respect to α on every subinterval $[c,d]$ of $[a,b]$. □

The next theorem shows that the HS integral satisfies the additive property on subintervals.

Theorem 3.5. If a function f is HS-integrable with respect to α on $[a,c]$ and $[c,b]$, then f is HS-integrable with respect to α on $[a,b]$. Moreover,

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

Proof. Let $\varepsilon > 0$. Suppose $\int_a^c f d\alpha = A_1$ and $\int_c^b f d\alpha = A_2$. Then there exists a positive function δ_1 on $[a,c]$ such that

$$\Big|\Big|S(f,\alpha;Q) - A_1\Big|\Big|_p < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever Q is a δ_1-fine tagged partition of $[a,c]$ and there exists a positive function δ_2 on $[c,b]$ such that

$$\Big|\Big|S(f,\alpha;R) - A_2\Big|\Big|_p < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever R is a δ_2-fine tagged partition of $[c,b]$. Define a positive function δ on $[a,b]$ by

$$\delta(x) = \begin{cases} min\{\delta_1(x), c-x\}, & if\ a \leq x < c \\ min\{\delta_1(c), \delta_2(c)\}, & if\ x = c \\ min\{\delta_2(x), x-c\}, & if\ c < x \leq b. \end{cases}$$

Let P be a δ-fine tagged partition of $[a,b]$. Note that

$$P = P_a \bigcup \{(c,[u,v])\} \bigcup P_b$$

where the tags of P_a are less than c and the tags of P_b are greater than c. Let $P_1 = P_a \bigcup \{(c,[u,c])\}$ and let $P_2 = P_b \bigcup \{(c,[c,v])\}$. Then P_1 is a δ_1-fine tagged partition of

$[a, c]$ and P_2 is a δ_2-fine tagged partition of $[c, b]$. Since $P = P_1 \bigcup P_2$, we have $S(f, \alpha; P) = S(f, \alpha; P_1) + S(f, \alpha; P_2)$. Hence,

$$\begin{aligned} \big|\big|S(f, \alpha; P) - (A_1 + A_2)\big|\big|_p &= \big|\big|S(f, \alpha; P_1) + S(f, \alpha; P_2) - (A_1 + A_2)\big|\big|_p \\ &\leq 2^{\frac{1}{p}}\Big(||S(f, \alpha; P_1) - A_1||_p\Big) + 2^{\frac{1}{p}}\Big(||S(f, \alpha; P_2) - A_2||_p\Big) \\ &< \varepsilon. \end{aligned}$$

□

The following theorem shows the linearity of HS integral.

Theorem 3.6. Let $k \in \mathbb{R}$.

1. If f is HS-integrable with respect to α on $[a, b]$, then kf is HS-integrable with respect to α on $[a, b]$. Moreover,
$$\int_a^b kf d\alpha = k \int_a^b f d\alpha.$$

2. If f and g are HS-integrable with respect to α on $[a, b]$, then $f + g$ is HS-integrable with respect to α on $[a, b]$. Moreover,
$$\int_a^b (f + g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha.$$

Proof. (1) Let f be HS-integrable with respect to α on $[a, b]$. The case $k = 0$ is obvious. Suppose $k \neq 0$. Since f is HS-integrable, there exists a positive function δ on $[a, b]$ such that
$$\big|\big|S(f, \alpha; P) - \int_a^b f d\alpha\big|\big|_p < \frac{\varepsilon}{|k|}$$
whenever P is a δ-fine tagged partition of $[a, b]$. Then,

$$\begin{aligned} \Big|\Big|S(kf, \alpha; P) - k \int_a^b f d\alpha\Big|\Big|_p &= \Big|\Big|kS(f, \alpha; P) - k \int_a^b f d\alpha\Big|\Big|_p \\ &< |k| \left(\frac{\varepsilon}{|k|}\right) = \varepsilon. \end{aligned}$$

(2) Let $\varepsilon > 0$. Suppose $\int_a^b f d\alpha = A_1$ and $\int_a^b g d\alpha = A_2$. Then there exists a positive function δ_1 on $[a, b]$ such that

$$\big|\big|S(f, \alpha; P_1) - A_1\big|\big|_p < \frac{\varepsilon}{2\Big(2^{\frac{1}{p}}\Big)}$$

whenever P_1 is a δ_1-fine tagged partition of $[a,b]$. Also, there exists a positive function δ_2 such that

$$||S(g,\alpha;P_2) - A_2||_p < \frac{\varepsilon}{2\left(2^{\frac{1}{p}}\right)}$$

whenever P_2 is a δ_2-fine tagged partitions of $[a,b]$. Define a positive function δ on $[a,b]$ by $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$ for all $x \in [a,b]$. Let P be a δ-fine tagged partition of $[a,b]$. Then

$$\begin{aligned} ||S(f+g,\alpha;P) - (A_1+A_2)||_p &= ||S(f,\alpha;P) + S(g,\alpha;P) - (A_1+A_2)||_p \\ &\leq 2^{\frac{1}{p}}\left(||S(f,\alpha;P) - A_1||_p + ||S(g,\alpha;P) - A_2||_p\right) \\ &< 2^{\frac{1}{p}}\left(\frac{\varepsilon}{2(2^{\frac{1}{p}})} + \frac{\varepsilon}{2(2^{\frac{1}{p}})}\right) = \varepsilon. \end{aligned}$$

□

Theorem 3.7. Let $k \in \mathbb{R}$.

1. If f is HS-integrable with respect to α on $[a,b]$, then f is HS-integrable with respect to $k\alpha$ on $[a,b]$. Moreover,
$$\int_a^b f d(k\alpha) = k\int_a^b f d\alpha.$$

2. If f is HS-integrable with respect to α and β on $[a,b]$, then f is HS-integrable with respect to $\alpha+\beta$ on $[a,b]$. Moreover,
$$\int_a^b f d(\alpha+\beta) = \int_a^b f d\alpha + \int_a^b f d\beta.$$

Proof. (1) Suppose that f is HS-integrable with respect to α on $[a,b]$. If $k=0$, then we are done. Suppose $k \neq 0$. Then there exists a positive function δ on $[a,b]$ such that

$$||S(f,\alpha;P) - \int_a^b f d\alpha||_p < \frac{\varepsilon}{|k|}$$

whenever P is a $\delta-$fine tagged partition of $[a,b]$. Now

$$\begin{aligned} \left|\left|S(f,k\alpha;P) - k\int_a^b f d\alpha\right|\right|_p &= \left|\left|kS(f,\alpha;P) - k\int_a^b f d\alpha\right|\right|_p \\ &< |k|\left(\frac{\varepsilon}{|k|}\right) = \varepsilon. \end{aligned}$$

(2) Suppose $\int_a^b f d\alpha = A_1$ and $[a,b]$ and let $\int_a^b f d\beta = A_2$. Then there exists a positive function δ_1 on $[a,b]$ such that

$$||S(f\alpha; P_1) - A_1||_p < \frac{\varepsilon}{2(2)^{\frac{1}{p}}}$$

whenever P_1 is a δ_1-fine tagged partition of $[a,b]$. Also, there exists a positive function δ_2 on $[a,b]$ such that

$$||S(f,\beta; P_2) - A_2||_p < \frac{\varepsilon}{2(2)^{\frac{1}{p}}}$$

whenever P_2 is a δ_2-fine tagged partition of $[a,b]$. Now, define δ on $[a,b]$ by $\delta(x) = min\{\delta_1(x), \delta_2(x)\}$ for all $x \in [a,b]$. Let P be any δ-fine tagged partition of $[a,b]$. Hence,

$$\begin{aligned} ||S(f,\alpha+\beta;P) - (A_1 + A_2)||_p &= ||S(f,\alpha;P) + S(f,\beta;P) - A_1 - A_2||_p \\ &\leq 2(2)^{\frac{1}{p}}||S(f,\alpha;P) - A_1||_p + 2(2)^{\frac{1}{p}}||S(f,\beta;P) - A_2||_p \\ &< 2(2)^{\frac{1}{p}}\left(\frac{\varepsilon}{2(2)^{\frac{1}{p}}}\right) + 2(2)^{\frac{1}{p}}\left(\frac{\varepsilon}{2(2)^{\frac{1}{p}}}\right) = \varepsilon \end{aligned}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

Abiac F. and Kalton, N. J. (2006). *Topics in Banach Space Theory*. Springer.

Boccuto A. and Sambucini, A. R. (2004). The Henstock-Kurzweil Integral for Functions Defined on Unbounded Intervals and with Values in Banach Spaces. *Acta Mathematica 7*, 3-17.

Carothers, N. L. (2005). *A Short Course on Banach Space Theory*. Cambridge University Press.

Cao, S. S. (1992). The Henstock Integral for Banach-Valued Functions. *SEA Bull. Math. 16*(1).

Giles, J. R. (2000). *Introduction to the Analysis of Normed Linear Spaces*, Cambridge University Press.

Gordon, R. A. (1994). *The Integrals of Lebesgue, Denjoy, Perron, and Henstock.* American Mathematical Society.

Kurtz, D. S. and Swartz, C. W. (2004). *Theories of Integration.* World Scientific.

Lim, J. S., Yoon, J. and Eun, G. (1998). On Henstock-Stieltjes Integral. *Kangweon-Kyungki Math. Journal 6*(1), 87-96.

Paluszynski M. and Stempak, K. (2009). On Quasi-Metric and Metric Spaces. *Proceeding of the American Mathematics Society 137*(12), 4307-4312.

Royden, H. L. (1968). *Real Analysis, 2nd Edition.* The Macmillan Company, New York.

Solodov, A. P. (1999). Henstock and McShane Integrals for Banach-Valued Functions. *Mathematical Notes 65*(6).